On Chip-firing in Infinite k-ary Trees

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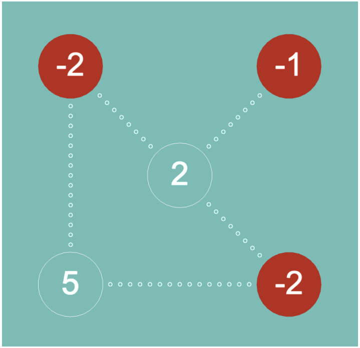
Contents

The Dollar Game

- One-player game played on a graph.
- Each vertex has a certain amount of "dollars" (that can be negative).
- On each turn, the player selects a vertex that will "give" a dollar to all its neighboring vertices.
- The goal is to make all the vertices have a nonnegative amount of dollars.

Dollar Game Example

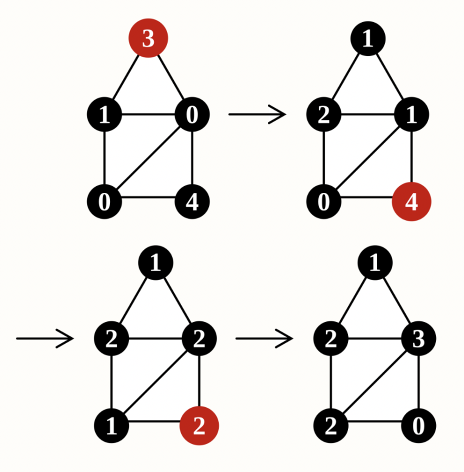
- Example



Chip-firing

- Similar 1 player “game”
- Each vertex has a certain number of indistinguishable chips, similar to chips in poker except these chips don’t have value.
- The number of chips at each vertex must always be nonnegative.
- We say a vertex is fired when we fire the vertex which gives one chip to each of its neighbors.
- **Goal of the game: find the stable configuration, such that no more vertices can be fired.**

Example of Chip-Firing on a Graph

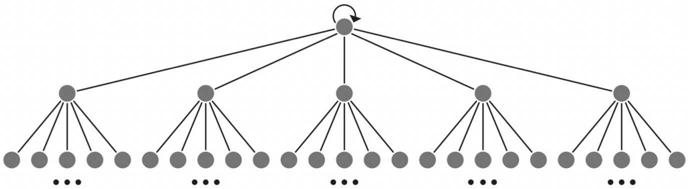


Abelian Sandpiles

- In an Abelian sandpile model, each part of a finite grid has a random number of chips.
- The slopes of chips eventually increase as chips are randomly placed on the grid.
- When the slope reaches a certain height, it collapses and splits its sand grains evenly among its neighbors.
- Chip-firing is similar to Abelian sandpiles, except with manual firing.

Our Underlying Graph

- A tree is a connected, undirected graph containing no cycles.
- A infinite perfect k-ary tree is a tree where each vertex has k children and the tree extends forever.
- Our underlying graph is a infinite perfect k-ary tree with a self-loop at the root.
- The self-loop makes it so that every vertex has $k + 1$ neighbors.
- Any number of chips can be placed at the root.

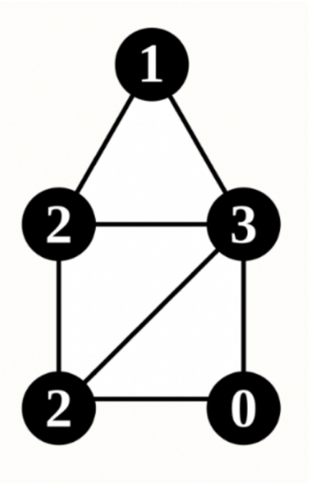


Past Work

- Past researchers used the same underlying graph with more restrictions.
- In 2023, Musiker and Nguyen found a formula for the number of fires on an infinite binary tree with a self-loop at the root, with $2^k - 1$ chips at the root initially.
- In 2024, Inagaki, Khovanova, and Lou generalized this to any number of chips at the root initially.

The Stable Configuration

- A stable configuration in chip-firing is a configuration of the graph such that none of the vertices can legally be fired.
- The graph to the right is in a stable configuration because every vertex has more connected neighbors than it has chips.
- Firing any vertex would result in a vertex with a negative number of chips.



Will our graph have a stable configuration?

The stable configuration is extremely important in chip-firing.

Theorem (Klivans, 2019). Either a stable configuration can be achieved after a finite number of fires or a stable configuration cannot be achieved.

Theorem (Björner–Lovász–Shor, 1991). If the number of chips is less than the number of edges, then the game is finite and will reach a stable configuration.

- In our case, since we have an infinite k-ary tree, the number of edges is infinite, and we have a finite number of chips at the root. Thus, the game is finite and reaches a stable configuration.

Stable Configuration of our Tree

Theorem (Klivans, 2019). The stable configuration is unique.

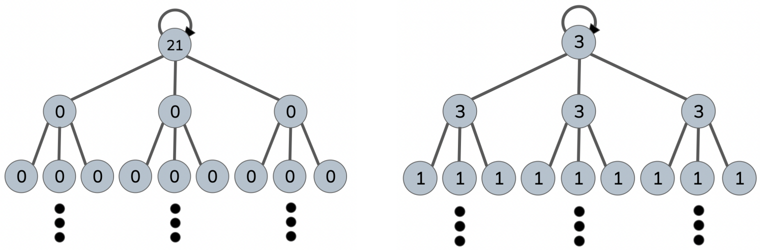
Proposition (A-G-G-K-K-M-P-P-R-S-X, 2025). If we start with N chips at the root, where

$$\frac{k^n - 1}{k - 1} \leq N \leq \frac{k^{n+1} - k}{k - 1},$$

then the vertices containing chips in the stable configuration form a perfect k-ary tree with height $n - 1$.

Furthermore, every vertex on the same layer has the same number of chips.

Example Stable Configuration



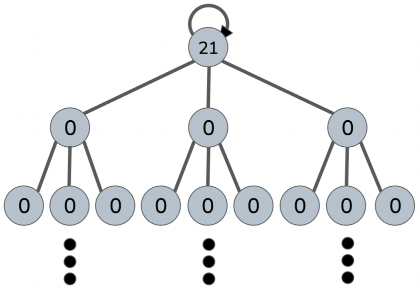
- The tree on the left has its stable configuration shown on the right.
- All numbers on each level are equal.
- They form a perfect ternary tree with 3 layers.

Reaching the Stable Configuration

- We can use the following steps to reach the stable configuration:
- 1. Fire the root repeatedly until it cannot fire anymore.
- 2. Fire the root's children and their subtrees in parallel.
- 3. Whenever the k children of the root fire, fire the root.
- 4. Repeat the second and third steps on subtrees until we reach the stable configuration.

Example

We will show the process that makes the previous tree obtain its stable configuration.



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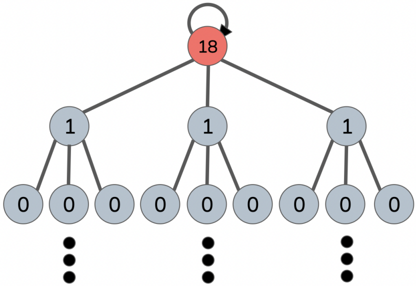
Fire the root repeatedly until it cannot fire anymore.

Fire the root's children and their subtrees in parallel.

Whenever the k children of the root fire, fire the root.

Repeat the second and third steps until we reach the stable configuration.



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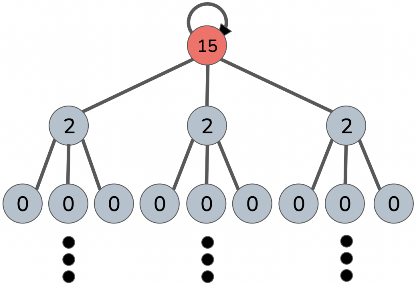

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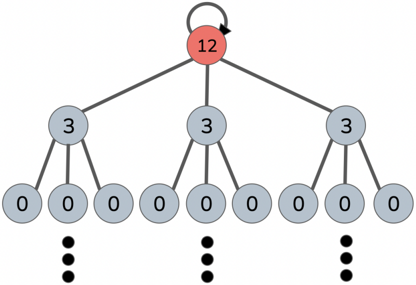

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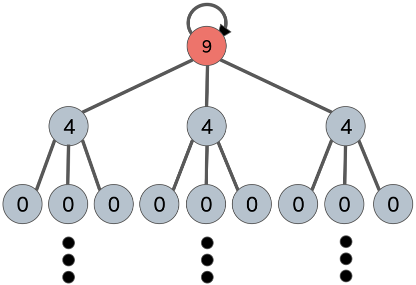

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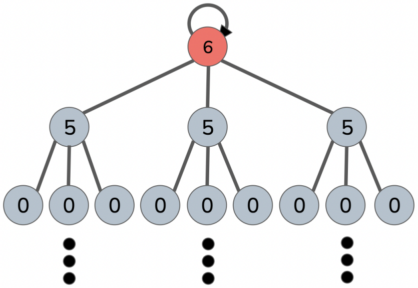

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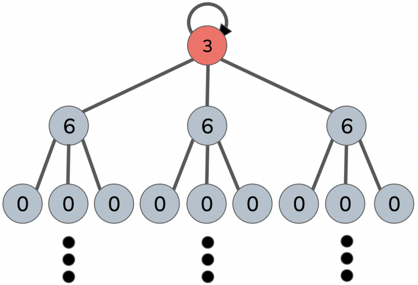

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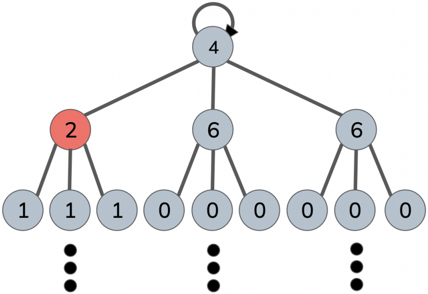

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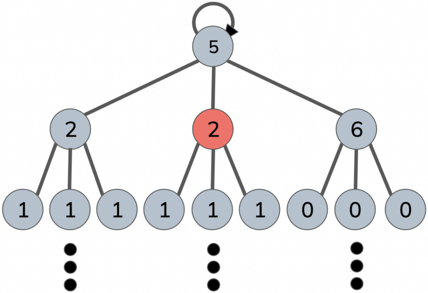

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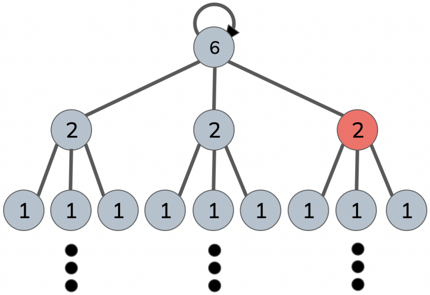

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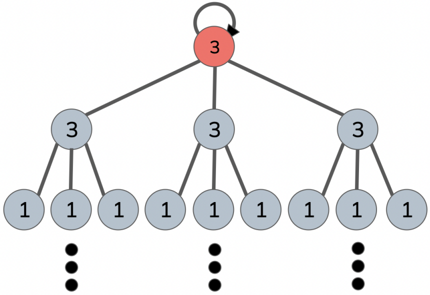
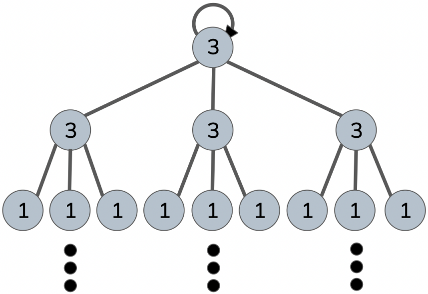


Chips per Layer

Proposition (A-G-G-K-K-M-P-P-R-S-X, 2025). If we start with N chips at the root, where $\frac{k^n-1}{k-1} \leq N \leq \frac{k^{n+1}-k}{k-1}$, then for $0 \leq i \leq n-1$, the resulting stable configuration has $a_i + 1$ chips on each vertex on layer $i+1$, where $a_{n-1} \ldots a_2 a_1 a_0$ is the base k expansion of $N - \frac{k^n-1}{k-1}$ with possible leading zeros.

- Each vertex has between 1 and k chips.
- Each vertex in the same layer has the same number of chips.
- This is very similar to base k.

Chips per Layer Example



In our example, where $N = 21$ inside a ternary tree, we take the base 3 expansion of $21 - \frac{3^3-1}{3-1} = 8$, which is $022_{(3)}$. This matches with the stable configuration, which is $1, 3, 3$ from the bottom up.

Example: Number of Fires in Each Layer

- In the previous example of 21 chips at the root of a ternary tree, the root fires 7 times and each of its children fire once.
- We define $f_i(N, k)$ as the number of fires on layer $i + 1$.
- We let $c_i(N, k)$ be the number of chips on a vertex in layer $i + 1$ in the stable configuration.
- The layer number of a vertex is one more than its distance to the root; in particular, the layer number of the root is 1.

Root Fires

Theorem (A-G-G-K-K-M-P-P-R-S-X, 2025). The number of root fires can be expressed as:

$$f_0(N,k) = \frac{1}{k-1}\sum_{j=1}^{n-1}(k^j-1)c_j(N,k),$$

where $n = \lfloor \log_k(N(k-1)+1) \rfloor$.This can be transformed into the following recursive formula:

$$f_0(N) = \left\lceil \frac{N}{k} \right\rceil - 1 + f_0\left(\left\lceil \frac{N}{k} \right\rceil - 1 \right)$$

Calculating number of fires per layer

- The difference in the number of fires between vertices on layers $i+1$ and $i+2$ equals the number of chips in the stable configuration belonging to the descendants of one particular vertex on layer $i+1$ divided by k.

Theorem (A-G-G-K-K-M-P-P-R-S-X, 2025). The difference number of fires on layer i can be expressed as:

$$f_i(N,k) - f_{i+1}(N,k) = \sum_{j=i+1}^{n-1} k^{j-i-1} c_j(N,k)$$

where $n = \lfloor \log_k(N(k-1)+1) \rfloor$.

Direct Formula

- Using induction, we can turn this formula to a direct formula.
  Theorem (A-G-G-K-K-M-P-P-R-S-X, 2025). The difference between the number of fires on layer i and layer $i+1$ can also be written as:

$$f_i(N,k) = \sum_{j=1}^{n-i-1} \left(\frac{k^j - 1}{k-1}\right) c_{i+j}(N,k).$$

Calculating the number of fires per layer

We can also express the number of fires of each vertex on layer $i+1$ through the number of fires of the root.

Corollary.

$$f_i(N,k) = f_0\left(\left\lfloor \frac{N - \frac{k^n-1}{k-1}}{k^i} \right\rfloor + \frac{k^{n-i}-1}{k-1}, k\right),$$

where $n = \lfloor \log_k(N(k-1)+1) \rfloor$.

Total Number of Fires

- Total number of fires is equal to the sum of the number of root fires and the sum of the fires of each of the k subtrees.
- This gives the following recursive formula:

Theorem. (A-G-G-K-K-M-P-P-R-S-X, 2025). The total number of fires expressed as the sum of fires of the root and its k subtrees is:

$$F(N) = f_0(N) + kF\left(\left\lceil \frac{N}{k} \right\rceil - 1\right).$$

- Each subtree behaves like the original tree with $\left\lceil \frac{N}{k} \right\rceil - 1$ chips.

Fires per Layer

- Notice that for all $N \in \{ak+1, ak+2, \ldots, (a+1)k\}$, the same vertex of the tree fires the same number of times.
- Thus, it makes sense to consider the number of fires as a function of $\lceil \frac{N}{k} \rceil$. Therefore, we introduce a new set of functions:

$$g_i(m,k) = f_i(mk,k).$$

Fires per Layer Table

- The table below shows the number of fires on layer 0 given m and k for $1 \le m \le 10$ and $2 \le k \le 6$.

$k \backslash m$	1	2	3	4	5	6	7	8	9	10
2	0	1	2	4	5	7	8	11	12	14
3	0	1	2	3	5	6	7	9	10	11
4	0	1	2	3	4	6	7	8	9	11
5	0	1	2	3	4	5	7	8	9	10
6	0	1	2	3	4	5	6	8	9	10

Table: Values of $g_0(m, k)$ for $1 \le m \le 10$ and $2 \le k \le 6$.

Difference sequence

- Consider the difference sequence $d_i(m,k) = g_i(m+1,k) - g_i(m,k)$. We are focused mainly on when $i = 0$.
- We derived a recursive sequence for $i = 0$:

$$d_0(m,k) = \begin{cases} d_0(\frac{m-1}{k}, k) + 1, & \text{if } m-1 \text{ is a multiple of } k, \\ 1, & \text{otherwise.} \end{cases}$$

Difference sequence (extended)

- Using the recursive sequence, we can now derive a formula.
- We define $\nu_k(x)$ as the maximum integer y such that $k^y | x$.

Theorem. (A-G-G-K-K-M-P-P-R-S-X, 2025). Term $d_0(m, k)$ in the difference sequence can be expressed as:

$$d_0(m,k) = \begin{cases} n, & \text{if } m = \frac{k^n - 1}{k-1} \text{ for some integer } n \\ \nu_k((k-1)m+1)+1, & \text{otherwise.} \end{cases}$$

Table for total fires

- We can generalize the previous statements for total fires.
- Similar to $f_i(N, k)$, notice that if $N \in \{ak + 1, ak + 2, \dots, (a + 1)k\}$, then the root and the other vertices of the tree fire the same number of times. Thus, it makes sense to consider the number of fires as a function of $\left\lceil \frac{N}{k} \right\rceil$. Therefore, we introduce a new function:

$$G(m, k) = F(mk, k).$$

Table for total fires (extended)

- The table below shows the total number of fires table for small values of m and k.

$k \backslash m$	1	2	3	4	5	6	7	8	9	10
2	0	1	2	6	7	11	12	23	24	28
3	0	1	2	3	8	9	10	15	16	17
4	0	1	2	3	4	10	11	12	13	19
5	0	1	2	3	4	5	12	13	14	15
6	0	1	2	3	4	5	6	14	15	16

Difference sequence for total fires

- We define the difference function as

$$D(m,k) = G(m+1,k) - G(m,k)$$

- We find a recursive formula for $D(m,k)$.

$$D(m,k) = \begin{cases} d_0(m) + kD(\frac{m-1}{k}), & \text{if } m-1 \text{ is a multiple of } k, \\ 1, & \text{otherwise.} \end{cases}$$

- Using the above recursive formula, we have the following theorem

$$D(m,k) = G(m+1,k) - G(m,k) = a(d_0(m,k),k).$$

Table for difference sequence

- The table below shows the total number of fires for small values of m and k.

$k \backslash m$	1	2	3	4	5	6	7	8	9	10
2	1	1	4	1	4	1	11	1	4	1
3	1	1	1	5	1	1	5	1	1	5
4	1	1	1	1	6	1	1	1	6	1
5	1	1	1	1	1	7	1	1	1	1
6	1	1	1	1	1	1	8	1	1	1

Tables for the Unique Values

- The table below shows the unique values from the previous table for consecutive differences between the total number of fires.

$k \backslash m$	1	2	3	4	5	6	7	A#
2	1	4	11	26	57	120	247	A000295
3	1	5	18	58	179	543	1636	A000340
4	1	6	27	112	453	1818	7279	A014825
5	1	7	38	194	975	4881	24412	A014827
6	1	8	51	310	1865	11196	67183	A014829
7	1	9	66	466	3267	22875	160132	A014830
8	1	10	83	668	5349	42798	342391	A014831
9	1	11	102	922	8303	74733	672604	A014832
10	1	12	123	1234	12345	123456	1234567	A014824

Tables for the Unique Values (extended)

- Each of these sequences are in the OEIS. Notably, when $k = 2$, the sequence corresponds to the Eulerian numbers.
- All these sequences have the same recursive formula:

$$a(n, k) = k \cdot a(n-1, k) + n.$$

- In base 10, the numbers look like the numbers from 1 to n concatenated, but this pattern fails after the 9th term.

Schizophrenic Numbers

- A schizophrenic number is an irrational number that has properties similar to rational numbers.
- This is described in László Tóth's paper on schizophrenic numbers.
- Taking the square root of numbers in our table results in schizophrenic numbers.
- For example, consider $a(11, 10) = 12345679011$. Then, the square root of this starts as

$$111111.1111050555555553905416666576734097216095565928.$$

- There are large sequences of repeating digits, a characteristic of schizophrenic numbers
- A comment on OEIS by Peter Bala claimed that the inverse of schizophrenic numbers also have patterns similar to schizophrenic numbers.

Examples

- When we go to larger terms, we can see the large patterns of schizophrenic numbers.
- For example, the square root of $a(19, 10) = 1234567901234567899$, starts as:

$$1111111111.11111111010555555555555555510054166666666 6.$$

- We can also see that the lengths of consecutive blocks of the same digits are decreasing, and after a while the digits seem random again.
- Taking the reciprocal of the square root of $a(19, 10)$, we get the number:

$$0.000000000900000000000000000814500000000000001105 6837.$$

- This also has similar properties as schizophrenic numbers, which gives motivation for Peter Bala's comment.

Schizophrenic Numbers in Different Bases

- Note: when we take the square root of the terms in the other rows of our table (i.e., when $k \neq 10$), we do not get schizophrenic numbers.
- But when we put them in base k, we get schizophrenic numbers.
- László Tóth's paper showed this table of taking the square root of the numbers on the fifth row in our unique values table and putting them in base 5.
- Here, $f_5(n)$ is $a(n, 5)$ in our tables.

n	$\sqrt{f_5(n)}$
7	$1111.11020303013402123214233234430313200224213102 40_5$
9	$11111.111010303030100244100302243334320304302441412_5$
11	$111111.111100030303030003021324330340130443133340 32_5$
13	$1.111111111104403030303024401244102132010133224210 2_5 \times 5^6$
15	$1.11111111111104303030303030242410213244102013310 02_5 \times 5^7$
17	$1.1111111111111104203030303030302412124410213244033_5 \times 5^8$
19	$1.1111111111111111041030303030303030234430213244102_5 \times 5^9$
21	$1.1111111111111111110400303030303030303023310244102_5 \times 5^{10}$
23	$1.1111111111111111111103403030303030303030302311402_5 \times 5^{11}$

Acknowledgements

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End of Presentation

Thank you so much for listening to our presentation!

Any questions?